

Two classes of quadratic vector fields for which the Kahan discretization is integrable

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1 INTRODUCTION

Kahan's method for discretizing quadratic differential equations was introduced in [1]. It was rediscovered in the context of integrable systems by Hirota and Kimura [2]. Suris and collaborators extended the applications to integrable systems significantly in a series of papers [3], [4], [5], [6], [7]. Applications to non-integrable Hamiltonian systems and the use of polarisation to discretise arbitrary degree Hamiltonian systems were studied in [8], [9] and [10].

The present paper contains an extract of the talk one of the authors (GRWQ) gave on 6th July 2016 at the 12th International Conference on Symmetries and Integrability of Difference Equations (SIDE12) in Sainte Adele, Quebec, Canada. We present two classes of 2-dimensional ODE systems of quadratic vector fields where the Kahan discretization is integrable. Both classes of systems can be cast in the form

$$\frac{dX}{dt} = \varphi(X)K\nabla H(X), \quad (1)$$

where

$$X^t = (x, y), \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $\varphi(X)$ is a scalar function of the components of X . In the first class the Hamiltonian function is quartic and in the second class it is sextic. These systems can be seen as generalisations of the examples of the reduced Nahm equations presented in [6]. Some of the results in this paper were found independently by Petrera and Zander [11].

2 Quartic Hamiltonians in 2D

Consider the 2-dimensional ODE system (1) where $\varphi(X) = \frac{1}{ax+by}$, and the homogeneous Hamiltonian has the form $H = (ax + by)^2(cx^2 + 2dxy + ey^2)$. Then the Kahan map for this system preserves the modified Hamiltonian:

$$\tilde{H}(X) = \frac{H}{(1 + h^2 D(ax + by)^2 + h^2 E(cx^2 + 2dxy + ey^2))(1 + h^2 9D(ax + by)^2 + h^2 E(cx^2 + 2dxy + ey^2))},$$

and the measure:

$$m(x, y) = \frac{dxdy}{(ax + by)(cx^2 + 2dxy + ey^2)}.$$

Here, $D := ce - d^2$ and $E := 2abd - a^2e - b^2c$. It follows that the Kahan map is integrable. It also seems to preserve the genus of the level sets of H (which generically equals 1).

3 Sextic Hamiltonians in 2D

Consider the 2-dimensional ODE system (1) with $\varphi(X) = \frac{1}{(cx+dy)(ex+fy)^2}$ and with the *homogeneous* sextic Hamiltonian $H = (ax + by)(cx + dy)^2(ex + fy)^3$. Then the Kahan map for this system preserves the modified Hamiltonian:

$$\tilde{H}(X) = \frac{H}{(1 + a_5 l_2^2)(1 + a_3 l_1^2 + a_4 l_3^2 + a_7 l_1 l_3)(1 + a_5 l_2^2 + a_6 l_3^2)}$$

where

$$l_1 := ax + by, \quad l_2 := cx + dy, \quad l_3 := ex + fy,$$

and $d_{1,2} := h(ad - bc)$, $d_{2,3} := h(cf - ed)$, $d_{3,1} := h(eb - fa)$ and $a_3 := \frac{-9d_{2,3}^2}{4}$, $a_4 := \frac{-d_{1,2}^2}{4}$, $a_5 := \frac{-9d_{3,1}^2}{4}$; $a_6 := -4d_{1,2}^2$, $a_7 := \frac{3d_{1,2}d_{2,3}}{2}$. In this case the Kahan map also preserves the modified measure

$$m(x, y) = \frac{dxdy}{(ax + by)(cx + dy)(ex + fy)}.$$

Again, the Kahan map is integrable. It also seems to preserve the genus of the level sets of H (which generically equals 1).

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